Yang-Mills theory. Fadeev-Popov quantization

$U(1)$ gauge theory in 3+1 d $\quad (-,+,+,+)$

- massless fermion $\quad \mathcal{L}_{fer} = -\bar{\psi} \not{\partial} \psi$

  $U(1)$ global symmetry $\quad \psi(x) \to e^{-i\varepsilon} \psi(x) \quad \delta \mathcal{L}_{fer} = 0$

  local symmetry $\quad \psi(x) \to e^{-i\varepsilon(x)} \psi(x)$

  $$\delta \mathcal{L}_{fer} = +i \bar{\psi} \gamma^\mu \psi \, \partial_\mu \varepsilon(x) \quad \text{no longer inv.}$$

- local / gauge invariant Lagrangian

  $$\mathcal{L}_{fer} = -\bar{\psi} \gamma^\mu (\partial_\mu + i A_\mu) \psi$$

  connection / gauge field

  $$A_\mu \to A_\mu + \partial_\mu \varepsilon$$

  $$\delta \mathcal{L}_{fer} = -i \bar{\psi} \gamma^\mu \psi \, \partial_\mu \varepsilon(x) + i \bar{\psi} \gamma^\mu \psi \, \partial_\mu \varepsilon(x) = 0$$

  $$\partial_\mu \to D_\mu = \partial_\mu + i A_\mu$$

  covariant derivative

  $$D_\mu \psi \to e^{-i\varepsilon(x)} D_\mu \psi$$

- gauge invariant Lagrangian: $\quad \mathcal{L}(\psi, \partial\psi) \to \mathcal{L}(\psi, D\psi)$

Lagrangian for A_μ? $\quad A = A_\mu dx^\mu \quad$ connection 1-form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad F = dA \quad \text{curvature}$$

$$F_{\mu\nu} \to F_{\mu\nu} \qquad \mathcal{L}_{U(1)} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$$

matter fields with non-Abelian gauge symmetry

- Lie group G, Lie algebra $\mathfrak{g}$, (unitary) representation R

  Ex $SU(2)$ $su(2)$ fund.

  elements in $\mathfrak{g}$ $\theta^a T_a$ (anti-Hermitian)

  elements in G $e^{\theta^a T_a}$ unitary

- fermionic matter $\psi^i : \mathbb{R}^{3,1} \to R \otimes \text{spinor}$

  i: index of R

$$\psi^i \longrightarrow (e^{-\lambda^a T_a})^i{}_j \psi^j$$

infinitesimal $\psi^i \longrightarrow \psi^i - \lambda^a (T_a)^i{}_j \psi^j$

- G invariant Lagrangian

  $\bar{\psi}$ in $\bar{R}$, $\bar{R} \otimes R = 1 \oplus \cdots$

$$\mathcal{L}_{matter} = -\bar{\psi}_i \not{\partial} \psi^i - m \bar{\psi}_i \psi^i$$

$\bar{\psi} = (\psi^i)^\dagger{}_i \gamma^0$

$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$

$\gamma_\mu^\dagger = \gamma^\mu$

G is the global symmetry of $\mathcal{L}$

local transformation : $\psi^i \to \psi^i - \lambda^a(x) (T_a)^i{}_j \psi^j$

$\mathcal{L}_{matter}$ is no longer invariant

define $\partial_\mu \longrightarrow D_\mu$

$$(D_\mu \psi)^i = \partial_\mu \psi^i + A_\mu^a (T_a)^i{}_j \psi^j$$

aim $\quad (D_\mu \psi)^i$ transforms as ψ^i

$$-\lambda^a(x)(T_a)^i{}_j D_\mu \psi^j = \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{-\lambda^a(x)(T_a)^i{}_j \partial_\mu \psi^j} - \lambda^a(x)(T_a)^i{}_j A^b_\mu T_b{}^j{}_k \psi^k$$

$$\|$$

$$\delta D_\mu \psi^i = \partial_\mu \delta\psi^i + A_\mu^a (T_a)^i{}_j \delta\psi^j + \delta A_\mu^a (T_a)^i{}_j \psi^j$$

$$= -\partial_\mu(\lambda^a(x)(T_a)^i{}_j \psi^j) - A_\mu^a (T_a)^i{}_j \lambda^b(x)(T_b)^j{}_k \psi^k$$

$$+ \delta A_\mu^a (T_a)^i{}_j \psi^j$$

$$= -(T_a)^i{}_j \psi^j \partial_\mu \lambda^a(x) \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{- \lambda^a(x)(T_a)^i{}_j \partial_\mu \psi^j}$$

$$- A_\mu^a \lambda^b(x) (T_a)^i{}_j (T_b)^j{}_k \psi^k + \delta A_\mu^a (T_a)^i{}_j \psi^j$$

$$\Rightarrow \delta A_\mu^a (T_a)^i{}_j \psi^j = (\partial_\mu \lambda^a(x)) (T_a)^i{}_j \psi^j$$

$$+ A_\mu^a \lambda^b(x) (T_a)^i{}_j (T_b)^j{}_k \psi^k$$

$$- A_\mu^a \lambda^b(x) (T_b)^i{}_j (T_a)^j{}_k \psi^k$$

$$\Rightarrow \delta A_\mu^a (T_a)^i{}_j \psi^j = (\partial_\mu \lambda^a)(T_a)^i{}_j \psi^j + A_\mu^a \lambda^b \underbrace{[T_a, T_b]^i{}_j}_{f_{ab}{}^c (T_c)^i{}_j} \psi^j$$

$$\Rightarrow \delta A_\mu^a = \partial_\mu \lambda^a + f_{bc}{}^a A_\mu^b \lambda^c$$

result: $\begin{cases} \delta\psi^i = -\lambda^a (T_a)^i{}_j \psi^j \\ \delta A_\mu^a = \partial_\mu \lambda^a + f_{bc}{}^a A_\mu^b \lambda^c \end{cases} \Rightarrow \begin{array}{l} \delta D_\mu \psi^i = -\lambda^a (T_a)^i{}_j D_\mu \psi^j \\ D_\mu \psi^i \longrightarrow (e^{-\lambda^a T_a})^i{}_j D_\mu \psi^j \end{array}$

Note: * $\mathcal{L}(\psi, \partial\psi)$ invariant Lagrangian under global G transf

$\mathcal{L}(\psi, D\psi)$ invariant Lagrangian under global G transf

* Jacobi identity

$$0 = [[T_a, T_b], T_c] + [[T_c, T_a], T_b] + [[T_b, T_c], T_a]$$

constraints of $f_{ab}{}^c$

$$0 = f_{ab}{}^d f_{dc}{}^e + f_{ca}{}^d f_{db}{}^e + f_{bc}{}^d f_{da}{}^e$$

also $f_{ab}{}^c = - f_{ba}{}^c$

* $\left.\begin{array}{l} T_a\text{'s anti-Hermitian} \\ [T_a, T_b] = f_{ab}{}^c T_c \end{array}\right\} \Rightarrow f_{ab}{}^c$ real → real Lie algebra

* simple Lie algebra

no proper ideal: no subset $\{L_a\}$, s.t $[L_a, T_b] \in \{L_a\}$ for some T_b

Semi-simple: direct sum of simple $\oplus\, u(1)$'s

* Killing form:

$$\hat{K}(X, Y) \equiv Tr(ad X\, ad Y) \qquad X, Y \in \mathfrak{g}$$

semisimple: $\hat{K}$ is nondegenerate

can be normalized as $K(T_a, T_b) = \delta_{ab}$

used to low $f_{ab}{}^c$ to f_{abc}

for semi-simple f_{abc} totally anti-symmetric

* adjoint rep

$$(T_a^A)_b{}^c = f_{ba}{}^c$$

$$[T_b^A, T_c^A]_a{}^e = f_{ab}{}^d f_{dc}{}^e - f_{ac}{}^d f_{db}{}^e = f_{bc}{}^d f_{ad}{}^e$$

$$= f_{bc}{}^d (T_d^A)_a{}^e$$

$$\delta A_\mu^a = \partial_\mu \lambda^a + f_{bc}{}^a A_\mu^b \lambda^c = \partial_\mu \lambda^a - \lambda^c f_{cb}{}^a A_\mu^b$$

$$= \partial_\mu \lambda^a - \lambda^c (T_b^A)_c{}^a A_\mu^b$$

$$= \partial_\mu \lambda^a + A^b{}_\mu (T_b^A)^a{}_c \lambda^c \equiv D_\mu \lambda^a \quad (\text{semi simple})$$

* $(A_\mu)^i{}_j = A_\mu^a (T_a)^i{}_j \qquad (D_\mu)^i{}_j = \partial_\mu \delta^i_j + (A_\mu)^i{}_j$

finite transformation rule for A_μ^a ?

$$(D_\mu \psi)^i \longrightarrow (e^{-\lambda^a T_a})^i{}_j (D_\mu \psi)^j$$

$$= (e^{-\lambda^a T_a})^i{}_j (D_\mu)^j{}_k (e^{\lambda^a T_a})^k{}_l (e^{-\lambda^a T_a})^l{}_m \psi^m$$

$$\Rightarrow D_\mu \longrightarrow e^{-\lambda^a T_a} D_\mu e^{\lambda^a T_a}$$

YM theory

$\mathcal{L}_{YM}$ for A^a_μ s.t inv. under local transf?

connection to curvature

$$[D_\mu , D_\nu] = [\partial_\mu + A_\mu , \partial_\nu + A_\nu]$$

$$= \underbrace{[\partial_\mu , \partial_\nu]}_{0} + [A_\mu , \partial_\nu] + [\partial_\mu , A_\nu] + [A_\mu , A_\nu]$$

$$[\partial_\mu , A_\nu] = \partial_\mu A_\nu - A_\nu \partial_\mu = (\partial_\mu A_\nu)$$

$$[A_\mu , \partial_\nu] = A_\mu \partial_\nu - \partial_\nu A_\mu = -(\partial_\nu A_\mu)$$

$$[A_\mu , A_\nu] = A_\mu^a A_\nu^b [T_a , T_b] = A_\mu^a A_\nu^b f_{ab}{}^c T_c$$

$$[D_\mu , D_\nu] = (\partial_\mu A_\nu) - (\partial_\nu A_\mu) + A_\mu^a A_\nu^b f_{ab}{}^c T_c$$

$$\equiv F_{\mu\nu} = F_{\mu\nu}^a T_a$$

curvature $\quad F_{\mu\nu} \equiv [D_\mu , D_\nu]$

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}{}^a A_\mu^b A_\nu^c$$

form: $\quad A = A_\mu dx^\mu \quad$ connection 1-form

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu \equiv dA + A \wedge A$$

curvature 2-form

Note: * in Abelian case $f_{ab}{}^{c} = 0$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad F = dA$$

* $F_{\mu\nu}$ contains 1-derivative of A_μ

$F^2 \sim$ kinetic term

transformation rule of $F^a_{\mu\nu}$

$$D_\mu \longrightarrow e^{-\lambda^a T_a} D_\mu e^{\lambda^a T_a}$$

$$\Rightarrow \quad F_{\mu\nu} \longrightarrow e^{-\lambda^a T_a} F_{\mu\nu} e^{\lambda^a T_a}$$ ← $F_{\mu\nu}$ transf gauge covariantly

infintesimal verision

$$\delta F_{\mu\nu} = -\lambda^a T_a F_{\mu\nu} + F_{\mu\nu} \lambda^a T_a$$
$$= \lambda^a F^b_{\mu\nu} [T_b, T_a] = -\lambda^a F^b_{\mu\nu} f_{ab}{}^{c} T_c$$

check:

use $\delta A^a_\mu = \partial_\mu \lambda^a + f_{bc}{}^{a} A^b_\mu \lambda^c$

and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^{a} A^b_\mu A^c_\nu$

to derive the above transf. rule.

Note:

GR		YM
$\Gamma^\rho_{\mu\nu}$, $\underset{\text{spin connection}}{\omega_\mu{}^a{}_b}$	$\sim$	$(A_\mu)^i{}_j$
$R_{\mu\nu\sigma}{}^{\rho}$, $R_{\mu\nu}{}^{a}{}_{b}$	$\sim$	$(F_{\mu\nu})^i{}_j$

Lorentz invariant quadratic combination

$$(F_{\mu\nu}F^{\mu\nu})_{ij}, \quad (\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma})_{ij}$$

gauge invariant combination from $F_{\mu\nu}F^{\mu\nu}$

$$\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} = F^a_{\mu\nu}F^{b,\mu\nu}\,\mathrm{tr}\, T_a T_b$$

$\mathrm{tr}\, T_a T_b$: Killing form

normalization $\mathrm{tr}\, T_a T_b = \frac{1}{2}\delta_{ab}$

$$\mathcal{L}_{YM} = -\frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a,\mu\nu}$$

In general, $\mathcal{L}_{YM} = -\frac{1}{2g^2}g_{ab}F^a_{\mu\nu}F^{b,\mu\nu}$ to be gauge inv.

requires: $g_{ab}F^a_{\mu\nu}f_{cd}{}^{b}F^{c,\mu\nu} = 0$
together with $g_{ab} = g_{ba}$

$$\Rightarrow \quad g_{ab}f_{cd}{}^{b} = -g_{cb}f_{ad}{}^{b}$$

also $\mathcal{L}_{YM}$ has to be positive def for quadratic terms $((\partial^0 A_\mu)^2 \ldots)$

$\Rightarrow$ g_{ab} is a real symmetric positive-definite matrix

w/ $g_{ab}f_{cd}{}^{b} = -g_{cb}f_{ad}{}^{b}$

or equivalent. Lie algebra is semi-simple

(proof: S. Weinberg 15.A)

Example: $SU(2)$ $\quad T_a = \frac{1}{2}\sigma_a \quad g_{ab} = \frac{1}{2}\delta_{ab}$

$$f_{abc} = \varepsilon_{abc}$$

topological term $\varepsilon^{\mu\nu\rho\sigma} \mathrm{tr} F_{\mu\nu} F_{\rho\sigma} \sim \mathrm{tr} F \wedge F \equiv \mathrm{tr} F^2$

$$F = dA + A \wedge A$$

$$\Rightarrow dF = dA \wedge A \underset{\uparrow}{-} A \wedge dA = [dA, A] = -[A, F]$$

$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ α is a p-form

$$\Rightarrow DF = dF + [A, F] = 0 \quad \text{Bianchi identity}$$

$$D_{[\mu} F_{\nu\rho]} = 0$$

$$d\,\mathrm{tr} F^2 = 2\,\mathrm{tr}\, dF\, F = -2\,\mathrm{tr}[A, F] F = 0$$

$\mathrm{tr} F^2$ is a closed 4-form. locally $\mathrm{tr} F^2 = d\omega_3$

but for polynomials $\mathrm{tr} F^2 = d\omega_3$ holds globally

Q: what's the form of ω_3

$$S_{top} = \theta \int_{M_3} \mathrm{tr} F^2 \quad \text{topological}$$

$\downarrow$

Chern form

- $$\mathcal{L}_{YM} = -\frac{1}{4g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} + \theta \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr} F_{\mu\nu} F_{\rho\sigma}$$

  - invariant under local transformation
  - rescale A_μ to remove $\frac{1}{g^2}$ in front of $F_{\mu\nu}F^{\mu\nu}$ but move g in D_μ
  - $\mathcal{L}_{YM}$ can't depends only on A^2. like mass term $-\frac{1}{2} m^2_{ab} A^a_\mu A^b_\nu$

equation of motion

$$\mathcal{L} = -\frac{1}{4g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_M(\psi, D_\mu \psi)$$

for now, no non-renormalizable terms like $D_\nu D_\mu \psi$, $D_\lambda F^a_{\mu\nu} \cdots$

$F_{\mu\nu} = F^a_{\mu\nu} T_a$ let $\operatorname{tr} T_a T_b$ normalized to δ_{ab}

$$\mathcal{L}_{YM}(A^a_\mu) = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a,\mu\nu}$$

$$= -\frac{1}{4g^2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu\right)^2$$

gauge invariant $\mathcal{L}_{YM}$ contains interaction $A^2 \partial A$, A^4

EOM: $$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a{}_\nu)} - \frac{\partial \mathcal{L}}{\partial A^a_\nu} = 0$$

$\Rightarrow D_\mu F^{a,\mu\nu} = -J^{a,\nu}$ $\qquad J^{a,\nu} \equiv -\frac{\partial \mathcal{L}}{\partial D_\nu \psi} T_a \psi$

$\Rightarrow D_\nu J^{a,\nu} = 0$ (compute $[D_\nu, D_\mu] F^{a,\rho\sigma}$ first)

hint: use the trick $(D_\mu F_{\rho\sigma})^a T_a = [D_\mu, F_{\rho\sigma}]$

Bianchi identity $D_{[\mu} F^a{}_{\nu\lambda]} = 0$

physical configuration:

solution of eom up to gauge equivalence

i.e. A^a_μ's related by gauge transf $\rightarrow$ same physical config.

different from global transf.

Faddeev - Popov quantization of YM theory

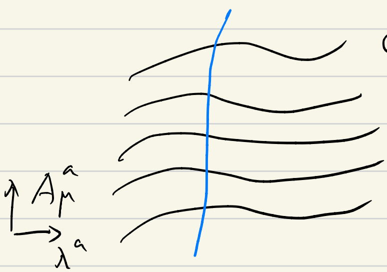
path-integral : Consider only YM action

$$I = \int \mathcal{D}A^a_\mu(x)\, e^{iS_{YM}} = \int \mathcal{D}A^a_\mu(x)\, e^{i\int d^4x\, \mathcal{L}_{YM}}$$

measure : $\mathcal{D}A^a_\mu = \prod_{a,\mu,x} dA^a_\mu(x)$

- path-integral over gauge orbits : redundancy (a lot!)
  remove redundancy : gauge fixing
- unphysical modes from $-(\partial_0 A_0)^2$: wrong sign
  remove by gauge transf.
  physical : transversal , unphysical : longitudial, temperal

modify path-integral : pick-up a "representative" in each gauge orbit



$$I \overset{?}{=} \int \mathcal{D}A^a_\mu(x)\, \delta[f^a[A^a_\mu]]\, e^{iS_{YM}} \quad ?$$

$f^a[A^a_\mu]$: gauge fixing term , the end result should be independent of the choice of f

Faddeev - Popov quantization

Convention: $A^a_{\mu,\lambda}(x)$ is the gauge transf. of $A^a_\mu(x)$ under the gauge transf. parameterized by $\lambda^a(x)$

$f^a[A^a_{\mu,\lambda}(x)]$ is implicitly functional of $\lambda^a(x)$

① $$1 = \int \mathcal{D}f^a \delta[f^a] = \int \mathcal{D}\lambda^a(x)\, \delta[f^a[A^a_{\mu,\lambda}(x)]] \det \frac{\delta f^a[A^a_{\mu,\lambda}(x)]}{\delta \lambda^b(y)}\Big|_{A^a_{\mu,\lambda}}$$

② $$I = \int \mathcal{D}A^a_\mu\, e^{iS_{YM}}\, 1 = \int \mathcal{D}A^a_\mu\, e^{iS_{YM}} \int \mathcal{D}\lambda^a(x)\, \delta[f^a] \det \frac{\delta f^a}{\delta \lambda^b(x)}\Big|_{A^a_{\mu,\lambda}}$$

$$= \int \mathcal{D}\lambda^a(x) \underbrace{\int \mathcal{D}A^a_\mu\, e^{iS_{YM}[A^a_\mu]}}_{\text{gauge inv.}} \delta[f^a] \det \frac{\delta f^a}{\delta \lambda^b(x)}\Big|_{A^a_{\mu,\lambda}}$$

$$= \int \mathcal{D}\lambda^a(x) \boxed{\int \mathcal{D}A^a_{\mu,\lambda}\, e^{iS_{YM}[A^a_{\mu,\lambda}]} \delta[f^a[A^a_{\mu,\lambda}]] \det \frac{\delta f^a}{\delta \lambda^b}\Big|_{A^a_{\mu,\lambda}}}$$

$A^a_{\mu,\lambda}$ is a dummy "index" here

③ "Shakespeare theorem" $\int dx\, f(x) = \int dx'\, f(x')$

"... O be some other name. What's in a name? that which we call a rose, by any other word would smell as sweete ..."

$$I = \underbrace{\int \mathcal{D}\lambda^a(x)}_{\substack{\text{an overall factor} \\ \text{"volume of the gauge orbit"}}} \underbrace{\int \mathcal{D}A^a_\mu\, e^{iS_{YM}[A^a_\mu]} \delta[f^a[A^a_\mu]] \det \frac{\delta f^a}{\delta \lambda^b}\Big|_{A^a_\mu(\lambda=0)}}_{\text{doesn't depend on } \lambda^a(x)}$$

in the end up to an overall factor $\int \mathcal{D}\lambda^a(x)$

$$I \propto \int \mathcal{D}A_\mu^a \, e^{iS_{YM}} \, \delta[f^a[A_\mu^a]] \det \frac{\delta f^a}{\delta \lambda^b}\Big|_{\lambda=0}$$

by derivation, the result is independent of gauge f^a

Compute $\delta[f^a]$ and $\det \frac{\delta f^a}{\delta \lambda^b}\Big|_{\lambda=0}$

- Lorentz gauge: $f^a[A_\mu^a] = \partial^\mu A_\mu^a$

$$\delta[f^a[A_\mu^a]] = e^{-\frac{i}{2\xi}\int d^4x \, f^a f^a} = e^{-\frac{i}{2\xi}\int d^4x \, (\partial^\mu A_\mu^a)^2}$$

$\det \frac{\delta f^a}{\delta \lambda^b}\Big|_{\lambda=0}$:

$$A^a_{\mu,\lambda}(x) = A_\mu^a(x) + \partial_\mu \lambda^a(x) + f_{bc}{}^a A_\mu^b(x)\lambda^c(x) + \cdots$$
$$= A_\mu^a(x) + D_\mu \lambda^a(x)$$

$$\frac{\delta f^a[A_\mu^a(x)]}{\delta \lambda^b(y)} = \partial^\mu \partial_\mu \delta^{ab} \delta^4(x-y) + \partial^\mu f_{cd}{}^a A_\mu^c(x) \delta^{db} \delta^4(x-y)$$

$$= \partial^\mu (D_\mu)^{ab} \delta^4(x-y)$$

(all derivatives are on x)

det? $\quad P_{\alpha\beta}$ a linear operator

bosonic path-integral $\quad \int \mathcal{D}\phi^\alpha \, e^{i\int d^4x \, \phi^\alpha P_{\alpha\beta} \phi^\beta} \sim \frac{1}{(\det P)^{1/2}}$

fermionic path-integral $\quad \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{i\int d^4x \, \bar{\psi}^\alpha P_{\alpha\beta} \psi^\beta}$

$$\sim \det P$$

ghost fields b_a, c^a : fermionic, scalar, adjoint fields

$$\det \frac{\delta f^a}{\delta \lambda^b} = \int \mathcal{D}b_a \mathcal{D}c^a \, e^{i\int d^4x \, b_a \partial^\mu D_\mu c^a}$$

$$= \int \mathcal{D}b_a \mathcal{D}c^a \, e^{-i\int d^4x (\partial^\mu b_a) D_\mu c^a}$$

$$D_\mu c^a = \partial_\mu c^a + f_{bc}{}^a A^b_\mu c^c$$

- general gauge, similar

$$S[f^a] = e^{-\frac{i}{2\xi}\int d^4x \, f^a f^a}$$

$$\det \frac{\delta f^a}{\delta \lambda^b} = \int \mathcal{D}b_a \mathcal{D}c^a \, e^{i\int d^4x \underline{\int d^4y} \, b_a(x) \frac{\delta f^a(x)}{\delta \lambda^b(y)} c^b(y)}$$

removed by $\delta^4(x-y)$ in $\frac{\delta f}{\delta \lambda}$

full quantum action

$$I = \int \mathcal{D}\lambda^a \int \mathcal{D}A^a_\mu \mathcal{D}b_a \mathcal{D}c^a \, e^{i\int d^4x \, \mathcal{L}_{qu}}$$

$$\mathcal{L}_{qu} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a,\mu\nu} - \frac{1}{2\xi}(f^a)^2 + \int d^4y \, b_a(x) \frac{\delta f^a(x)}{\delta \lambda^b(y)} c^b(y)$$

- Lorentz gauge $f^a = \partial^\mu A^a_\mu$

$$\mathcal{L}_{qu} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a,\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - (\partial^\mu b_a) D_\mu c^a$$

$\xi = 1$ 't Hooft - Feynman gauge

$\xi = 0$ Landau gauge

note: all derivations apply when we insert gauge inv. operators $\mathcal{O}_i$

$$\langle \mathcal{O}_i \cdots \rangle = \int \mathcal{D}\lambda^a \int \mathcal{D}A^a_\mu \mathcal{D}b_a \mathcal{D}c^a \, \mathcal{O}_i \, e^{i\int d^4x \, \mathcal{L}_{qu}}$$

Perturbative computation

use scalars as example

free scalars

infinitesimal Wick rotation

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + i\varepsilon\phi^2$$

add source term $-J\phi$: $\mathcal{L} = \mathcal{L}_0 - J\phi$

$$Z_{\mathcal{L}_0}[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x (\mathcal{L}_0 - J\phi)}$$

generating functional of Green functions

$$G(x_1 \cdots x_n)_{free} = \langle 0 | T[\textstyle\prod_i \phi(x_i)] | 0 \rangle = \langle T[\textstyle\prod_i \phi(x_i)] \rangle$$

$$= \prod_i \left(i\hbar \frac{\delta}{\delta J(x_i)} \right) Z_{\mathcal{L}_0}[J] \Big|_{J=0}$$

perform the Gaussian integral for $Z_{\mathcal{L}_0}[J]$

$$Z_0[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}\int d^4x \left[\frac{1}{2}\phi(\partial^\mu\partial_\mu - m^2 + i\varepsilon)\phi - J\phi\right]}$$

completing square by change of the variable

$$\hat{\phi}(x) = \hat{\phi}(x) - \int d^4y \, \Delta_F(x-y) J(y)$$

where $(\partial^\mu \partial_\mu - m^2 + i\varepsilon)\Delta_F(x) = \delta^4(x)$

$$\Rightarrow Z_{\mathcal{L}_0}[J] = \exp\left(-\frac{i}{2\hbar}\int d^4x \, d^4y \, J(x) \Delta_F(x-y) J(y)\right) Z_{\mathcal{L}_0}[0]$$

notice $i\hbar \frac{\delta}{\delta J(x)} Z_{\mathcal{L}_0}[J]$ bring one $\phi(x)$ down in the path-integral

$$i\hbar \frac{\delta}{\delta J(x)} Z_{\mathcal{L}_0}[J] = \int \mathcal{D}\phi(x) \, \phi(x) \, e^{\frac{i}{\hbar}\int d^4x (\mathcal{L}_0 - J\phi)}$$

interacting scalars

$$Z_{\mathcal{L}}[J] = \frac{\int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}\int d^4x \left(\mathcal{L}_0 - V(\phi) - J\phi\right)\right)}{\int \mathcal{D}\phi \, \exp\left(\frac{i}{\hbar}\int d^4x \left(\mathcal{L}_0 - V(\phi)\right)\right)} \begin{matrix} \to \tilde{Z}[J] \\ \\ \to \tilde{Z}[0] \end{matrix}$$

$Z_{\mathcal{L}}[J]$ is related to the free one $Z_{\mathcal{L}_0}[J]$ by

$$Z_{\mathcal{L}}[J] = \tilde{Z}[0]^{-1} \exp\left(-\frac{i}{\hbar}\int d^4x \, V\left(i\hbar \frac{\delta}{\delta J(x)}\right)\right) Z_{\mathcal{L}_0}[J]$$

correlators $\langle T[\prod_i \phi(x_i)]\rangle$

$$\langle T[\prod_i \phi(x_i)]\rangle = \frac{1}{\tilde{Z}[0]}\left(\prod_i i\hbar \, \delta_J(x_i)\right) Z_{\mathcal{L}}[J]\Big|_{J=0}$$

$$= \frac{1}{\tilde{Z}[0]}\left(\prod_i i\hbar \, \delta_J(x_i)\right) \exp\left(\frac{-i}{\hbar}\int dx \, V(i\hbar \, \delta_J(x))\right) Z_{\mathcal{L}_0}[J]\Big|_{J=0}$$

where $\delta_J(x) = \delta/\delta J(x)$

perturbation theory

- if $V(\phi)$ depends on small coupling λ, expand $\exp\left(-\frac{i}{\hbar}\int dx \, V\right)$ and compute $\langle T[\prod_i \phi(x_i)]\rangle$ order by order
- at each order, we have a couple of δ_J's, then expand $Z_{\mathcal{L}_0}[J]$ to find the term with same J's (in the end we set $J=0$)
- Feynman diagrams are a nice tool to keep track of δ_J's acting on J's

Green functions

$$G(x_1 \cdots x_k) = \frac{\langle 0 | T\left(\prod_{i=1}^{k} \phi(x_i) \exp\left(\frac{i}{\hbar}\int d^4y\, V(i\hbar\delta_J(y))\right)\right)|0\rangle_{free}}{\langle 0 | T\left(\exp\left(\frac{i}{\hbar}\int d^4y\, V(i\hbar\delta_J(y))\right)\right)|0\rangle_{free}}$$

- Ex. $\lambda\phi^3$ theory $\quad V = \frac{g}{3!}\phi^3$, compute $G(x_1\, x_2)$

$$G^{(0)}(x_1\, x_2) = (i\hbar)^2 \delta_J(x_1)\delta_J(x_2)\, Z_{\mathcal{L}_0}[J]\Big|_{J=0}$$

$$= (i\hbar)^2 \delta_J(x_1)\delta_J(x_2)\left(-\frac{i}{\hbar}\right)\int d^4w\int d^4z\, J(w)\Delta_F(w-z)J(z)$$

$$= \hbar\,(i\Delta_F(x_1-x_2))$$

$G^{(1)}(x_1\, x_2) = 0$ because $Z_{\mathcal{L}_0}[J]$ gives only even # of J's

$$G^{(2)}(x_1, x_2) = (i\hbar)^2\delta_J(x_1)\delta_J(x_2)\frac{1}{2}\left\{\frac{-i}{\hbar}\int d^4y\frac{g}{3!}(i\hbar\,\delta_J(y))^3\right\}^2$$

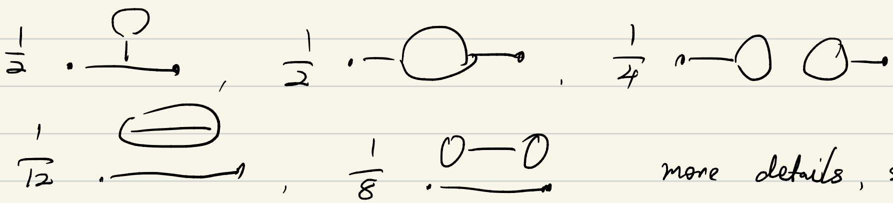
$$\times\frac{1}{4!}\left(\frac{-i}{2\hbar}\int d^4w\, d^4z\, J(w)\Delta_F(w-z)J(z)\right)^4 \longrightarrow \text{numerator}$$

$$-\Big((i\hbar)^2\delta_J(x_1)\delta_J(x_2)\frac{-i}{2\hbar}\int d^4w\, d^4z\, J(w)\Delta_F(w-z)J(z)\Big)$$

$$\times\frac{1}{2}\left\{\frac{g}{3!}\int d^4y\frac{-i}{\hbar}(i\hbar\,\delta_J(y))^3\right)^2 \longrightarrow \text{denominator}$$

$$\times\frac{1}{3!}\left(\frac{-i}{2\hbar}\int d^4w\, d^4z\, J(w)\Delta_F(w-z)J(z)\right)^3$$

then the problem is to solve the combinatoric problem of matching 8 δ_J's with 8 J's, in diagrams



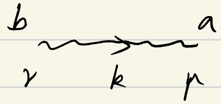
more details, see Sterman

Feynman rules

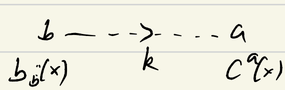
there is a summary of Feynman rules in the appendix of "gauge theory of elementary particle physics" by Cheng, Li

treat ghosts as new fields, combining all quadratic terms in $\mathcal{L}_{qn}$ to get propagators, other terms are interactions

propagators:

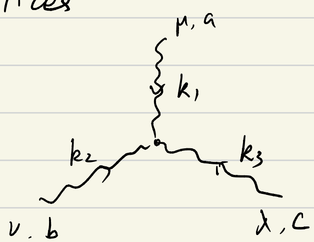
vector: 

$$\frac{-i\delta^{ab}}{k^2+i\varepsilon}\left[\eta_{\mu\nu}+(\xi-1)\frac{k_\mu k_\nu}{k^2}\right]$$

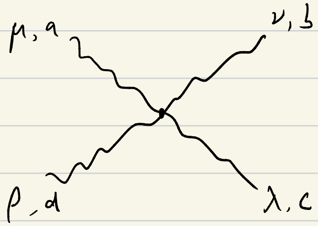
ghost: 

$$\frac{-i\delta^{ab}}{k^2+i\varepsilon}$$

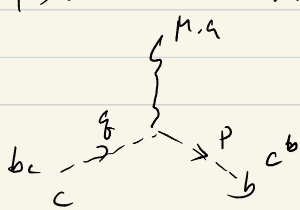
vertices



$$ig f_{abc}\left[(k_1-k_2)_\lambda\eta_{\mu\nu}+(k_2-k_3)_\mu\eta_{\nu\lambda}+(k_3-k_1)_\nu\eta_{\mu\lambda}\right]$$



$$\begin{aligned}-ig^2\big[&f_{eab}f_{ecd}(\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda})\\&+f_{eac}f_{edb}(\eta_{\mu\rho}\eta_{\lambda\nu}-\eta_{\mu\nu}\eta_{\lambda\rho})\\&+f_{ead}f_{ebc}(\eta_{\mu\nu}\eta_{\rho\lambda}-\eta_{\mu\lambda}\eta_{\rho\nu})\big]\end{aligned}$$



$$g f_{abc}\, p_\mu$$